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On the representations of $GL_{p,q}(2)$, $GL_{p,q}(1|1)$ and non-commutative spaces

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Abstract. Explicit realizations of the quantum groups $GL_{p,q}(2)$ and $GL_{p,q}(1|1)$ corresponding to unimodular values of the deformation parameters p and q are given in terms of the canoncially conjugate $(\hat{X}, \hat{P})$ operators, using the Heisenberg–Weyl commutation relations. Matrix representations are also discussed. Some observations are made on similar realizations of the non-commutative coordinate spaces on which the quantum groups act as endomorphisms.

1. Introduction

The mathematical structure of quantum groups (Drinfeld 1985, Jimbo 1986) has its origin in the study of quantum inverse scattering methods (Sklyanin *et al* 1979, Faddeev 1982) where the crux of the integrability lies in the quantum Yang–Baxter equation (Yang 1967, Baxter 1982). The quantum groups underlie the spectral parameter-independent limit of the trigonometric/hyperbolic solutions of the Yang–Baxter equation, whereas the classical Lie algebras correspond to its rational solutions. A quantum group and its Lie algebra may be viewed as deformations—generally depending on one or more parameters—of a classical Lie group and its universal enveloping algebra, with a comultiplication rule that preserves the defining commutation relations. The comultiplication is a generalization of the familiar tensor product of representations of classical groups. The close kinship between the Yang–Baxter algebras and the various physical and mathematical theories such as the analysis of braid groups and link invariants (Kauffman 1990, Kauffman and Saleur 1990) and the rational conformal field theory (Alvarez-Gaumé *et al* 1990) establishes their deep connection with the representations of the quantum algebras. The notion of non-commutative differential geometry (Connes 1985) underlies the quantum group structure and the related ideas are found to have interesting applications in string field and gauge theories (Witten 1986, 1990).

In the viewpoint proposed by Manin (1988, 1989) a quantum group is identified with endomorphisms acting on a non-commutative vector space, the Manin plane, the coordinates of which obey sets of bilinear product relations. The sufficient condition for the associativity of the algebra turns out to be the Yang–Baxter equation, the analogue of the Jacobi identity for the quantum groups. In the defining matrix representations of the endomorphisms the commutation relations for the non-commutative space coordinates generate the commutation relations to be satisfied by the elements of the quantum matrix. For the quantum group $GL_q(n)$, characterized by the standard single deformation (quantization) parameter q, the minimal set of commutation relations imposed by Manin's construction (Corrigan *et al* 1990) may be reduced (Floratos 1989, Weyers 1990) to the Heisenberg–Weyl form for unimodular values of q. Exploiting this property, explicit realization of the elements of the quantum group matrices may be obtained in terms of mutually commuting pairs of canonically conjugate $(\hat{X}, \hat{P})$ operators $(\hat{X}_j, \hat{P}_j | [\hat{X}_j, \hat{P}_k] = \mathrm{i}\delta_{jk}, [\hat{X}_j, \hat{X}_k] = [\hat{P}_j, \hat{P}_k] = 0)$ and matrices (Floratos 1989, Weyers 1990, Chakrabarti and Jagannathan 1991a).

Considering the quantum group co-acting on a pair of distinct quadratic spaces, Demidov *et al* (1990) (see also Sudbery 1990, Takeuchi 1990, Reshetikhin 1990, Kulish 1990, Schirrmacher *et al* 1991) generalized the Manin construction to obtain the non-standard deformation $\mathrm{GL}_{p,q,\varepsilon}(n)$ where p and q are two continuous parameters and ε is a finite set of ± 1 consistent with the functional independence of the generators in the sense of the Poincare–Birkhoff–Witt theorem. In this case the bilinear commutation relations imposed by the Manin construction may also be recast in terms of an R-matrix satisfying the Yang–Baxter equation. For $\mathrm{GL}_{p,q}(2)$ all the entries of ε may be chosen as 1. Developing the differential calculus on $\mathrm{GL}_{p,q}(2)$, leading to its Lie algebra, Schirrmacher *et al* (1991) emphasize that the significance of the two-parameter deformation comes to bear in the co-multiplication rule and the structure of the R-matrix, which truly depends on both parameters. The present authors recently obtained an oscillator realization of the corresponding quantum algebra (Chakrabarti and Jagannathan 1991b). The importance of the two-parameter quantum groups and their Lie algebras may be apparent in several contexts, as in the Yang–Baxterization procedure (Jones 1989, Ge and Xue 1991) to obtain the spectral parameter-dependent solutions of the Yang–Baxter equation, non-standard quantum statistics (Greenberg 1991) and various heuristic phenomenological applications such as in nuclear and molecular physics (Iwao 1990, Raychev *et al* 1990, Chang *et al* 1991). As pointed out by Schirrmacher *et al* (1991) the existence of a two-parameter deformation implies an infinite number of one-parameter deformations (in the standard case $p = q$), and it is interesting to study the consequence of this fact from the point of view of model building in physics (Chakrabarti and Jagannathan 1991c).

The purpose of the present paper is three-fold.

(i) We notice that the bilinear product relations of $\mathrm{GL}_{p,q}(2)$ may also be recast in the form

$$m_A m_B = p^{N_{AB}} q^{N'_{AB}} m_B m_A \tag{1.1}$$

where m_A $(A = 1\text{-}4)$ are the appropriate Heisenberg–Weyl variables. For unimodular values of p and q (we restrict ourselves to this case hereafter) we use the commutation relations (1.1) to obtain the realizations of the elements of the $\mathrm{GL}_{p,q}(2)$ matrix in terms of canonically conjugate $(\hat{X}, \hat{P})$ operators and matrices. The differential calculus on the quantum plane (Wess and Zumino 1990) is covariant under the quantum group and the application of the formalism of differential calculus on the non-commutative space of a quantum group (Woronowicz 1987) leads to the notion of quantization of the corresponding Lie algebra. For a concrete realization of the differential calculus in the space of a 'continuous' set of non-commuting variables, the given set of variables may be expressed as functions of a set of continuous numerical parameters. Floratos (1990) addressed this problem in the case of the Manin plane $A_q^{n|0}$ with the coordinates $(x_i | i = 1, 2, \ldots, n)$ satisfying the commutation relations $(x_i x_j = q^{-1} x_j x_i,\ i < j;\ i, j = 1, 2, \ldots, n)$ and gave a solution for unitary coordinates $(x_i^\dagger x_i = 1, |q| = 1)$ in terms of the quantum mechanical phase space operators $(\hat{X}, \hat{P})$ using the Heisenberg–Weyl relations. For the elements of the quantum matrices of $\mathrm{GL}_q(n)$ with $|q|=1$, a similar solution using the Heisenberg–Weyl relations is possible (Floratos 1989, Weyers 1990, Chakrabarti and Jagannathan 1991a). Here, we extend this solution to the case of $\mathrm{GL}_{p,q}(2)$. As noted by Floratos (1990), irreducible matrix representations do not facilitate the embedding of such a set of continuous numerical parameters in the non-commutative space as desired. We note that in this case a lattice of integer parameters can be introduced in such representations.

(ii) The natural generalization in the case of supergroups, corresponding to a single deformation parameter, is the quantum group $\mathrm{GL}_q(1|1)$ (Schwenk *et al* 1990, Schmidke *et al* 1990), i.e. the deformation of the supergroup of 2×2 non-singular matrices with two bosonic and two fermionic elements. The tools of the quantum inverse-scattering method may be developed in this case as there exists a universal R-matrix satisfying a Z_2-graded Yang–Baxter equation. An R-matrix satisfying the spectral parameter-dependent Yang–Baxter equation may be obtained and is known to be associated (Kauffman and Saleur 1990) with a picture of free fermions 'propagating' on the knot diagram and thereby realizing the Alexander–Conway polynomial as a Berezin integral. Here, we study the representation theory of the two-parameter quantum supergroup $\mathrm{GL}_{p,q}(1|1)$ defined by the endomorphisms acting on a pair of distinct superplanes (Manin 1989). The corresponding R-matrix which depends on the two deformation parameters and satisfies a Z_2-graded Yang–Baxter equation is exhibited. There exists a comultiplication rule similar to the one for $\mathrm{GL}_q(1|1)$. It is shown that, with a suitable choice of variables, the bilinear relations satisfied by the elements of the defining 2×2 matrix representation of $\mathrm{GL}_{p,q}(1|1)$ can again be translated into the Heisenberg–Weyl form and, consequently, their representations may be analysed in the same way as for $\mathrm{GL}_{p,q}(2)$.

(iii) Our treatment of the representation of the elements of the quantum matrices anchors on a 'factorization' procedure derived from earlier studies (Weyl 1950, Schwinger 1960, Ramakrishnan 1971, 1972, Jagannathan and Ranganathan 1974, 1975, Ramakrishnan and Jagannathan 1976, Jagannathan 1985) on generalized Clifford algebras for which the generators obey exactly the same relations as (1.1) apart from being non-singular. These algebras arise in the theory of projective representations of Abelian groups and have been studied in detail from the points of view of mathematical structure (see Morinaga and Nono 1952, Yamazaki 1964, Popovici and Gheorghe 1966, Morris 1967, 1973, Backhouse and Bradley 1972, and references therein) and physical applications (Weyl 1950, Schwinger 1960, Ramakrishnan 1972, Boon 1972, Jagannathan and Ranganathan 1976, Santhanam 1977, Jagannathan 1983, Baxter 1989, and references therein). In this context we outline briefly the work of Floratos (1990) on the representations of Manin's non-commutative coordinate space $A_q^{n|0}$ using our approach.

The plan of the paper is as follows. In sections 2 and 3 we discuss, in sequence, the Manin construction and the technique of deriving explicitly the realizations of $\mathrm{GL}_{p,q}(2)$ using the Heisenberg–Weyl relations. A parallel analysis is repeated in the same order for $\mathrm{GL}_{p,q}(1|1)$ in sections 4 and 5. In section 6 we analyse the representation of the Manin quantum plane and concluding remarks follow in section 7.

2. The Manin construction for $\mathrm{GL}_{p,q}(2)$

Following Demidov *et al* (1990) we first summarize the construction of $\mathrm{GL}_{p,q}(2)$ (for $pq\neq -1$, as we assume below) viewed as endomorphisms of a pair of distinct quadratic spaces,

$$A_p^{2|0} = \kappa\langle x, y\rangle/(xy - p^{-1}yx) \tag{2.1}$$

and the dual space

$$A_q^{*0|2} = \kappa\langle \zeta, \eta\rangle/(\zeta^2, \eta^2, \zeta\eta + q\eta\zeta). \tag{2.2}$$

The new feature is that the quantum space $A_p^{2|0}$ is generally distinct from $A_q^{2|0}$. Now, consider a matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_{p,q}(2) \tag{2.3}$$

which effects the simultaneous linear transformations

$$\begin{aligned} &M\colon A_p^{2|0} \to A_p^{2|0} \\ &M\colon A_q^{*0|2} \to A_q^{*0|2}. \end{aligned} \tag{2.4}$$

The elements of M are assumed to commute with the coordinates x, y, ζ and η. The endomorphisms (2.4) impose the following commutation relations on the elements of M;

$$\begin{aligned} &ab = q^{-1}ba \qquad cd = q^{-1}dc \qquad ac = p^{-1}ca \qquad bd = p^{-1}db \\ &bc = p^{-1}qcb \qquad [a, d] = (p^{-1} - q)cb. \end{aligned} \tag{2.5}$$

The quantum determinant

$$D(M) = ad - q^{-1}bc \tag{2.6}$$

is not central, but satisfies the commutation relations

$$aD = Da \qquad bD = p^{-1}qDb \qquad cD = pq^{-1}Dc \qquad dD = Dd. \tag{2.7}$$

The comultiplication rule for the matrix representation of $\mathrm{GL}_{p,q}(2)$, (2.3), preserves the commutation relations (2.5) and may be stated as follows: if $M, M' \in \mathrm{GL}_{p,q}(2)$ and the elements of M pairwise commute with the elements of M' then $MM' = M'' \in \mathrm{GL}_{p,q}(2)$. The quantum determinant follows the rule

$$D(M'') = D(M)D(M'). \tag{2.8}$$

Using the quantum determinant, the left and the right inverses may be defined as

$$M_{\mathrm{L}}^{-1} = D^{-1}\begin{pmatrix} d & -pb \\ -p^{-1}c & a \end{pmatrix} \tag{2.9}$$

and

$$M_{\mathrm{R}}^{-1} = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} D^{-1} \tag{2.10}$$

respectively, such that $M_{\mathrm{L}}^{-1}M = I$ and $MM_{\mathrm{R}}^{-1} = I$. As a consequence of the commutation relations (2.7) it follows that

$$M_{\mathrm{L}}^{-1} = M_{\mathrm{R}}^{-1} = M^{-1}, \text{say} \to M^{-1} \in \mathrm{GL}_{p^{-1},q^{-1}}(2). \tag{2.11}$$

The commutation relations (2.5) may be interpreted as the analogue of the symplectic conditions for $\mathrm{GL}_{p,q}(2)$. To this end, define

$$\varepsilon_q = \begin{pmatrix} 0 & q^{1/2} \\ -q^{-1/2} & 0 \end{pmatrix} \tag{2.12}$$

such that

$$\varepsilon_q^2 = -I. \tag{2.13}$$

Then, the commutation relations (2.5) can be stated equivalently as

$$M\varepsilon_q M^{\mathrm{T}} = \varepsilon_q D(M) \qquad M^{\mathrm{T}}\varepsilon_p M = \varepsilon_p D(M) \tag{2.14}$$

representing the analogue of the symplectic conditions for $\mathrm{GL}_{p,q}(2)$.

For any integer n it is easily seen that if $M \in \mathrm{GL}_{p,q}(2)$ then $M^n \in \mathrm{GL}_{p^n,q^n}(2)$. This may be shown as follows. Let

$$M^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \tag{2.15}$$

with the quantum determinant

$$D_n = a_n d_n - q^{-n} b_n c_n. \tag{2.16}$$

For all $m, n \in \mathbb{Z}$ the following relations hold:

$$\begin{aligned}
&a_n b_m - q^{-m} b_n a_m = -q^{-m} b_{n-m} D_m \qquad c_n d_m - q^{-m} d_n c_m = c_{n-m} D_m \\
&a_n c_m - p^{-n} c_n a_m = -p^{-n+m} D_m c_{n-m} \qquad b_n d_m - p^{-n} d_n b_m = p^{-m} D_m b_{n-m} \\
&p^n b_n c_m = q^m c_n b_m \qquad a_n d_m - p^{-n} q^{-m} d_n a_m = a_{n-m} D_m - p^{-n} q^{-m} d_{n-m} D_m \\
&a_n d_m - p^{-n} c_n b_m = a_{n-m} D_m \qquad d_n a_m - p^n b_n c_m = D_m d_{n-m} \qquad D_m = D^m.
\end{aligned} \tag{2.17}$$

Equations (2.17) can be proved by a double induction procedure: first an induction in n with $m = 1$ and then an induction in m with fixed n. In the limiting case $p = q$, (2.17) agree with the corresponding equations of Vokos *et al* (1990). For the case $n = m$ with $a_0 = d_0 = 1$ and $b_0 = c_0 = 0$, (2.17) reduce to

$$\begin{aligned}
&a_n b_n - q^{-n} b_n a_n = 0 \qquad c_n d_n - q^{-n} d_n c_n = 0 \\
&a_n c_n - p^{-n} c_n a_n = 0 \qquad b_n d_n - p^{-n} d_n b_n = 0 \\
&p^n b_n c_n - q^n c_n b_n = 0 \qquad [a_n, d_n] = (q^{-n} - p^n) b_n c_n \\
&a_n d_n - p^{-n} c_n b_n = D_n \qquad d_n a_n - p^n b_n c_n = D_n
\end{aligned} \tag{2.18}$$

proving the assertion that $M^n \in \mathrm{GL}_{p^n,q^n}(2)$ for any $n \in \mathbb{Z}$.

The bilinear product relations (2.5) may also be understood as the relations dictated by an R-matrix condition for the quantum group $\mathrm{GL}_{p,q}(2)$, namely

$$R_{i_1 i_2, k_2 k_1} M_{k_1 j_1} M_{k_2 j_2} = M_{i_2 k_2} M_{i_1 k_1} R_{k_1 k_2, j_2 j_1} \tag{2.19}$$

where the R-matrix is given by (Demidov *et al* 1990)

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 1-pq & q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.20}$$

with $(i_1 i_2)$ and $(k_1 k_2)$ labelling the rows and columns, respectively. The R-matrix is a linear transformation acting on a direct product space $V^{\otimes 2}$ and satisfies the Yang–Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{2.21}$$

as a sufficient condition for associativity; the notation R_{ij} denotes an operator acting on a triple tensor product of vector spaces $V_i \otimes V_j \otimes V_k$ such that its action on $V_i \otimes V_j$ is described by the R-matrix and its action on V_k reduces to the identity.

3. On the representations of the elements of $\mathrm{GL}_{p,q}(2)$ matrices

As already mentioned in the introduction, to get a concrete realization of the differential calculus on quantum matrices one may embed a set of continuous numerical parameters in the representation consistent with the commutation relations obeyed by the matrix elements. To show how this can be done in the case of $\mathrm{GL}_{p,q}(2)$ we extend the earlier work on $\mathrm{GL}_q(n)$ in this regard (Floratos 1989, Weyers 1990, Chakrabarti and Jagannathan 1991a).

Let us choose

$$m_1 = b \qquad m_2 = c \qquad m_3 = d \qquad m_4 = D(M) \tag{3.1}$$

and write

$$a = (D + q^{-1}bc)d^{-1} \tag{3.2}$$

assuming that d^{-1} exists. Then from (2.5) and (2.7) it follows that $(m_A) = (m_1, m_2, m_3, m_4)$ satisfy the bilinear product relations of the Heisenberg–Weyl form (1.1) for unimodular values of p and q. Taking, in general,

$$p = \exp(\mathrm{i}\chi) \qquad q = \exp(\mathrm{i}\lambda) \qquad 0 \leqslant \chi, \lambda < 2\pi \tag{3.3}$$

we have

$$m_A m_B = \exp(\mathrm{i}\phi_{AB}) m_B m_A \qquad \phi_{BA} = -\phi_{AB} \tag{3.4}$$

where the antisymmetric matrix $[\phi_{AB}] = \Phi$, say, is given by

$$\Phi = \begin{pmatrix} 0 & \lambda-\chi & -\chi & \lambda-\chi \\ \chi-\lambda & 0 & -\lambda & \chi-\lambda \\ \chi & \lambda & 0 & 0 \\ \chi-\lambda & \lambda-\chi & 0 & 0 \end{pmatrix} \tag{3.5}$$

with $\chi \neq \lambda$, $\det \Phi \neq 0$. Now, following Weyl (1950), one can write a unitary realization of (m_A) as

$$m_A \sim \exp\left(\mathrm{i} \sum_{B=1}^{4} u_{AB} \hat{Q}_B \right) \tag{3.6}$$

where $\sim$ denotes 'up to a constant multiplicative (normalization) factor', $(\hat{Q}_1, \hat{Q}_2, \hat{Q}_3, \hat{Q}_4) = (\hat{P}_1, \hat{X}_1, \hat{P}_2, \hat{X}_2)$ are such that

$$[\hat{Q}_A, \hat{Q}_B] = -\mathrm{i}\varepsilon_{AB} \qquad [\varepsilon_{AB}] = \varepsilon = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{3.7}$$

and $U=[u_{AB}]$ is any real matrix satisfying the relation

$$\Phi = U\varepsilon U^{\mathrm{T}}. \tag{3.8}$$

In the present case a solution for U, say $U^{(0)}$, is given by

$$U^{(0)}=\begin{pmatrix} (\chi-\lambda)\sigma_1 & \chi & (\chi^2-\lambda^2)^{1/2}\sigma_1 & 0 \\ (\chi-\lambda)\sigma_2 & \lambda & (\chi^2-\lambda^2)^{1/2}\sigma_2 & 0 \\ 1 & 0 & 0 & 0 \\ 1+(\chi\sigma_1-\lambda\sigma_2) & 0 & 0 & (\chi^2-\lambda^2)^{1/2} \end{pmatrix} \qquad \chi\sigma_2-\lambda\sigma_1=1 \tag{3.9}$$

as can be checked easily. Consequently, taking $[u_{AB}]=U=U^{(0)}$ in (3.6) we get a realization of (m_A) in terms of the quantum mechanical phase space operators:

$$\begin{aligned} &m_1 \sim \exp\{\mathrm{i}[(\chi-\lambda)\sigma_1\hat{P}_1+\chi\hat{X}_1+\sigma_1(\chi^2-\lambda^2)^{1/2}\hat{P}_2]\} \\ &m_2 \sim \exp\{\mathrm{i}[(\chi-\lambda)\sigma_2\hat{P}_1+\lambda\hat{X}_1+\sigma_2(\chi^2-\lambda^2)^{1/2}\hat{P}_2]\} \\ &m_3 \sim \exp(\mathrm{i}\hat{P}_1) \\ &m_4 \sim \exp\{\mathrm{i}[(1+\chi\sigma_1-\lambda\sigma_2)\hat{P}_1+(\chi^2-\lambda^2)^{1/2}\hat{X}_2]\} \\ &\chi\sigma_2-\lambda\sigma_1=1. \end{aligned} \tag{3.10}$$

If $S=[s_{AB}]$ is any 4×4 real matrix belonging to the symplectic group $\mathrm{Sp}(4,\mathbb{R})$, i.e. $S\varepsilon S^{\mathrm{T}}=\varepsilon$, then the linear canonical transformation $(\hat{Q}'_A=\Sigma_{B=1}^4 s_{AB}\hat{Q}_B \mid A=1,2,3,4)$ provides an equivalent $(\hat{P}'_1,\hat{X}'_1,\hat{P}'_2,\hat{X}'_2)$ which can replace $(\hat{P}_1,\hat{X}_1,\hat{P}_2,\hat{X}_2)$, respectively, in (3.10). In other words, one can have a continuous set of U-matrices $(U=U^{(0)}S \mid S\in\mathrm{Sp}(4,\mathbb{R}))$ providing the desired realizations of (m_A) through (3.6) independent of the actual representation of $\hat{Q}$s. Thus, the above (boson) realization of (m_A) (or the elements of the $\mathrm{GL}_{p,q}(2)$ matrix) through (3.1) and (3.2) with the facility to vary continuously a set of real parameters consistent with the prescribed commutation relations demonstrates how one may implement the formalism of differential calculus in the non-commutative space of the elements of the $\mathrm{GL}_{p,q}(2)$ matrix.

In the context of a similar realization of the Manin plane $A_q^{n|0}$, Floratos (1990) has noted that when q is a root of unity the irreducible representations are given by finite-dimensional matrices and it is not possible to embed in the above manner any set of continuous parameters consistent with the required commutation relations. Here also, we note that the above realization of $\mathrm{GL}_{p,q}(2)$ in terms of the $(\hat{X},\hat{P})$ operators is, in general, not irreducible with the matrix Φ being fixed for a given set of values for (p,q). It turns out that if we want to restrict ourselves to irreducible representations then it is possible to introduce only a discrete set of integer parameters in such representations instead of the set of continuous parameters, as in the above realization (3.10). To this end we shall examine the cases where (i) both p and q are roots of unity, and (ii) p and q are not roots of unity but are commensurate such that χ/λ is rational.

Let p and q be distinct roots of unity. Without loss of generality we may take

$$p=\exp(\mathrm{i}2\pi k/N) \qquad q=\exp(\mathrm{i}2\pi l/N) \qquad k\neq l \qquad k,l\in\mathbb{Z}_N. \tag{3.11}$$

Now, the commutation relations (3.4) become

$$m_Am_B=z^{n_{AB}}m_Bm_A \qquad n_{BA}=-n_{AB} \qquad z=\exp(\mathrm{i}2\pi/N) \tag{3.12}$$

with the antisymmetric integer matrix $[n_{AB}] = P$, say, given by

$$P = \begin{pmatrix} 0 & l-k & -k & l-k \\ k-l & 0 & -l & k-l \\ k & l & 0 & 0 \\ k-l & l-k & 0 & 0 \end{pmatrix}. \tag{3.13}$$

Analogous to (3.6) let us write (m_A) as a 'product transform' (Ramakrishnan and Jagannathan 1976) of a canonical set (μ_A):

$$m_A \sim \prod_{B=1}^{4} \mu_B^{\tilde{u}_{AB}} \tag{3.14}$$

where

$$\mu_A\mu_B = z^{\tilde{\varepsilon}_{AB}}\mu_B\mu_A \qquad [\tilde{\varepsilon}_{AB}] = \tilde{\varepsilon} = \begin{pmatrix} 0 & \varepsilon_1 & 0 & 0 \\ -\varepsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_2 \\ 0 & 0 & -\varepsilon_2 & 0 \end{pmatrix} \tag{3.15}$$

$$0 \leqslant \varepsilon_1,\ \varepsilon_2 < N-1$$

and $[\tilde{u}_{AB}] = \tilde{U}$ is any integer matrix with entries in $\mathbb{Z}_N$ such that

$$P = \tilde{U}\tilde{\varepsilon}\tilde{U}^T \ (\mathrm{mod}\ N). \tag{3.16}$$

It is easy to check that (m_A) defined by (3.14)–(3.16) have the correct commutation relations (3.12).

As is well known, for any antisymmetric integer matrix it is possible to have a decomposition of the form (3.16) with U as a unimodular integer matrix (here, $|\det \tilde{U}| = 1 \bmod N$) (e.g., see, Newman (1972) for an explicit algorithm to compute a unique $\tilde{\varepsilon}$ and a $\tilde{U}$ for a given P of any dimension). In the present case for P as given in (3.13), $\tilde{\varepsilon}$ and a solution for $\tilde{U}$, say $\tilde{U}^{(0)}$, are given by

$$\varepsilon_1 = \gcd(k, l) \qquad \varepsilon_2 = (k^2 - l^2)/\varepsilon_1$$

$$\tilde{U}^{(0)} = [\tilde{u}_{AB}^{(0)}] = \begin{pmatrix} (k-l)\tilde{\sigma}_1/\varepsilon_1 & k/\varepsilon_1 & \tilde{\sigma}_1 & 0 \\ (k-l)\tilde{\sigma}_2/\varepsilon_1 & l/\varepsilon_1 & \tilde{\sigma}_2 & 0 \\ 1 & 0 & 0 & 0 \\ 1+[(k\tilde{\sigma}_1 - l\tilde{\sigma}_2)/\varepsilon_1] & 0 & 0 & 1 \end{pmatrix} \tag{3.17}$$

$$k\tilde{\sigma}_2 - l\tilde{\sigma}_1 = \varepsilon_1 \ (\mathrm{mod}\ N) \qquad \tilde{\sigma}_1, \tilde{\sigma}_2 \in \mathbb{Z}_N.$$

Now, if we let

$$\omega_1 = z^{\varepsilon_1} \qquad \omega_2 = z^{\varepsilon_2} \tag{3.18}$$

then

$$\omega_1^{N_1} = \omega_2^{N_2} = 1 \qquad N_1 = N/\gcd(N, \varepsilon_1) \qquad N_2 = N/\gcd(N, |\varepsilon_2|). \tag{3.19}$$

The commutation relations (3.15) become

$$\begin{aligned} &\mu_1\mu_2 = \omega_1\mu_2\mu_1 \qquad \mu_3\mu_4 = \omega_2\mu_4\mu_3 \\ &\mu_A\mu_B = \mu_B\mu_A \text{ otherwise.} \end{aligned} \tag{3.20}$$

In terms of the μs satisfying (3.20) ms are given by the formula

$$m_A \sim \prod_{B=1}^{4} \mu_B^{\tilde{u}^{(0)}_{AB}} \qquad A=1\text{–}4. \tag{3.21}$$

Let us now recall that any irreducible representation of the relation

$$\mathbb{A}\mathbb{B} = \omega \mathbb{B}\mathbb{A} \qquad \omega^N = 1 \qquad \mathbb{A}^N = \mathbb{B}^N = 1 \tag{3.22}$$

with ω as a primitive Nth root of unity, is equivalent to the unitary N-dimensional representation (Weyl 1950):

$$\begin{gathered}
\mathbb{A} = h_N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix} \\
\mathbb{B} = g_N(\omega) = \begin{pmatrix} 1 & & & & \\ & \omega & & 0 & \\ & & \omega^2 & & \\ & 0 & & \ddots & \\ & & & & \omega^{N-1} \end{pmatrix}.
\end{gathered} \tag{3.23}$$

Hence, taking

$$\begin{aligned}
&\mu_1 = h_{N_1} \otimes 1 \qquad && \mu_2 = g_{N_1}(\omega_1) \otimes 1 \\
&\mu_3 = 1 \otimes h_{N_2} \qquad && \mu_4 = 1 \otimes g_{N_2}(\omega_2)
\end{aligned} \tag{3.24}$$

in (3.21) we have the only inequivalent irreducible representation of (m_A) by non-singular matrices, apart from the normalization factors. It may be noted that the irreducibility of (μ_A) implies the irreducibility of (m_A), and vice versa since the transformation $(\mu_A) \to (m_A)$ is invertible: $\tilde{\varepsilon} = \tilde{U}^{-1} P (\tilde{U}^{-1})^T$ and $\tilde{U}^{-1}$ is also an integer matrix with unit determinant mod N. Thus, when both p and q are roots of unity the irreducible representations of (m_A) by non-singular matrices are all finite-dimensional and hence the representation (3.6) is reducible.

It may be noted that the introduction of a set of continuous parameters in the realization (3.6) is made possible by the existence of the continuous group of linear canonical transformations of $(\hat{Q}_A)$. In the case of the finite-dimensional representation (3.21) the analogue is provided by the canonical product transformations of (μ_A): with $V = [v_{AB}]$, $v_{AB} \in \mathbb{Z}_N$,

$$\mu'_A \sim \prod_{B=1}^{4} \mu_B^{v_{AB}} \to \mu'_A \mu'_B = z^{\tilde{\varepsilon}_{AB}} \mu'_B \mu'_A \qquad \text{if } V\tilde{\varepsilon}V^{\mathrm{T}} = \tilde{\varepsilon} (\mathrm{mod}\ N). \tag{3.25}$$

Clearly the set of such V-matrices form a finite subgroup of $\mathrm{Sp}(4, \mathbb{R})$ and in (3.21) one may replace $\tilde{U}^{(0)}$ by a $(\tilde{U}^{(0)} V)$ consistent with the commutation relations for (m_A). Thus, when both p and q are roots of unity, if we want to restrict ourselves to irreducible representations, instead of the reducible representation (3.6), we can consider the elements of the $\mathrm{GL}_{p,q}(2)$ matrix only as functions of a set of integer variables $\in \mathbb{Z}_N$ independent of the actual matrix representations in the construction (3.21). Of course, the group of similarity transformations would provide a set of continuous parameters in any finite-dimensional representation, consistent with the algebraic relations, but it would be representation-dependent.

Let us now consider the case when $p=\exp(\mathrm{i}\chi)$ and $q=\exp(\mathrm{i}\lambda)$ are not roots of unity but χ/λ is rational. Then, one can write, in general,

$$p=z'^{k'} \qquad q=z'^{l'} \qquad k', l' \in \mathbb{Z} \qquad z'^{N} \neq 1 \text{ for any } N \in \mathbb{Z}. \qquad (3.26)$$

In this case the representation of (m_A) in terms of (μ_A) follows exactly as discussed above with the replacement of z, k and l by z', k' and l', respectively, and taking $\mathbb{Z}_N \rightarrow \mathbb{Z}$ since z' is not a root of unity. The resulting commutation relations for (μ_A) are

$$\begin{aligned} &\mu_1\mu_2 = z'^{\varepsilon_1'}\mu_2\mu_1 \qquad &&\mu_3\mu_4 = z'^{\varepsilon_2'}\mu_4\mu_3 \\ &\varepsilon_1' = \gcd(k', l') \qquad &&\varepsilon_2' = (k'^2 - l'^2)/\varepsilon_1'. \end{aligned} \qquad (3.27)$$

The commutation relation of the form $\mathbb{A}\mathbb{B}=\omega\mathbb{B}\mathbb{A}$ where ω is not a root of unity has been studied in detail earlier in physics literature in the context of the theory of Bloch electrons in an external homogeneous magnetic field. It turns out that in this case one has an infinite number of irreducible representations, all of infinite dimensions, and a representation of the form

$$\begin{aligned} &\mathbb{A} \sim \exp[\mathrm{i}(t_{11}\hat{P} + t_{12}\hat{X})] \qquad \mathbb{B} \sim \exp[\mathrm{i}(t_{21}\hat{P} + t_{22}\hat{X})] \\ &\exp[\mathrm{i}(t_{11}t_{22} - t_{12}t_{21})] = \omega \end{aligned} \qquad (3.28)$$

can be decomposed into irreducible constituents in an infinite number of different ways (for details, see Boon 1972). Hence, in the present case (μ_A), obeying (3.27), and (m_A) have an infinite number of irreducible representations, all of infinite dimensions, and the $(\hat{X}, \hat{P})$ operator realization (3.6) is decomposable into the irreducible constituents in an infinite number of different ways. Since the μs defined by (3.27) admit canonical product transformations of the form (3.25) with $V'\tilde{\varepsilon}'V^{\mathrm{T}} = \tilde{\varepsilon}'$, and $v'_{AB} \in \mathbb{Z}$, independent of the actual irreducible representation, in this case the elements of the $\mathrm{GL}_{p,q}(2)$-matrix can also be considered to be functions of a set of discrete variables $\in \mathbb{Z}$. It may be noted that when both p and q are roots of unity the representation (3.6) is decomposable into the finite-dimensional irreducible components uniquely (Boon 1972).

When p and q are not roots of unity and are also incommensurate (i.e. χ/λ is irrational) then it is not possible to obtain a representation formula like (3.14) expressing (m_A) as products of integral powers of more elementary building blocks, namely (μ_A). It is obvious that the irreducible representations have to be infinite-dimensional in this case. We may conjecture that in this case too the representation (3.6) will be decomposable in the same sense as discussed above for the case when p and q are not roots of unity but are commensurate.

In the representation considered above (m_A) are all invertible. But the ansatz (3.1), (3.2) requires only d, or m_3, to be invertible. This implies that when both p and q are roots of unity only μ_1 in (3.21) is necessarily invertible. This points to the possibility of other representations. For example, one may take

$$\mu_1 \sim \omega_1^{\pm J_0^{(1)}} \qquad \mu_2 \sim J_{\pm}^{(1)} \qquad \mu_3 \sim \omega_2^{\pm J_0^{(2)}} \qquad \mu_4 \sim J_{\pm}^{(2)} \qquad (3.29)$$

with

$$\omega_j^{J_0^{(j)}} J_{\pm}^{(j)} = \omega_j^{\pm 1} J_{\pm}^{(j)} \omega_j^{J_0^{(j)}} \qquad j=1,2 \qquad (3.30)$$

and $(J_0^{(1)}, J_\pm^{(1)})$ commuting with $(J_0^{(2)}, J_\pm^{(2)})$. Regarding (3.30), for each $j=1, 2$, as part of the generating relations of the su(2) algebra would lead to realizations in terms of $(\hat{X}, \hat{P})$ operators or boson creation and annihilation operators with the facility to embed a set of continuous parameters exactly as in the case of (3.6). The corresponding irreducible matrix representations can be obtained by regarding (3.30) as part of the generating relations of sl(2) or, more generally, $\mathrm{sl}_q(2)$ (Roche and Arnaudon 1989), and the set of integer parameters that may be embedded in such representations can be discussed for each specific representation. It is obvious that one can discuss similar realizations also in the case when p and q are not roots of unity but are commensurate (now, $\omega_1 = z'^{k'}$, $\omega_2 = z'^{l'}$); in contrast to the realization considered earlier where μs are regarded as the generators of projective representations of Abelian groups (Boon 1972), the representations of (3.29) and (3.30) can be finite-dimensional (Rosso 1988).

In the limiting case $p=q$, the above discussion contains as a special case the earlier results on $\mathrm{GL}_q(2)$ (Floratos 1989, Weyers 1990, Chakrabarti and Jagannathan 1991a).

4. The Manin construction for $\mathrm{GL}_{p,q}(1|1)$

In this section we consider the quantum supergroup $\mathrm{GL}_{p,q}(1|1)$, which may be viewed as the superanalogue of $\mathrm{GL}_{p,q}(2)$. The quantum supergroup $\mathrm{GL}_q(1|1)$ with a single deformation parameter has been previously studied in detail by several authors (Schwenk *et al* 1990, Schmidke *et al* 1990). To study $\mathrm{GL}_{p,q}(1|1)$, the two-parameter extension of $\mathrm{GL}_q(1|1)$, we shall follow the approach of Manin (1989) and Demidov *et al* (1990) (see also Soni 1991). So, we consider the endomorphisms of a pair of quadratic spaces with a bosonic and a fermionic variable;

$$A_p^{1|1} = \kappa\langle x, \eta\rangle/(\eta^2, x\eta - p^{-1}\eta x) \tag{4.1}$$

and the dual space

$$A_q^{*1|1} = \kappa\langle \zeta, y\rangle/(\zeta^2, \zeta y - qy\zeta). \tag{4.2}$$

Let

$$M = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \in \mathrm{GL}_{p,q}(1|1) \tag{4.3}$$

with its elements (anti)commuting with the coordinates of $A_p^{1|1}$ and $A_q^{*1|1}$. Then, the endomorphisms

$$\begin{aligned} M&: A_p^{1|1} \to A_p^{1|1} \\ M&: A_q^{*1|1} \to A_q^{*1|1} \end{aligned} \tag{4.4}$$

impose the following bilinear product relations among the elements of M:

$$\begin{aligned} &a\beta = q^{-1}\beta a \qquad a\gamma = p^{-1}\gamma a \qquad d\beta = q^{-1}\beta d \qquad d\gamma = p^{-1}\gamma d \\ &\beta^2 = \gamma^2 = 0 \qquad p\beta\gamma + q\gamma\beta = 0 \qquad [a, d] = (p^{-1} - q)\gamma\beta. \end{aligned} \tag{4.5}$$

The quantum superdeterminant of M may be defined as

$$\mathrm{Sdet}(M) = ad^{-1} - \beta d^{-1}\gamma d^{-1} \tag{4.6}$$

provided d^{-1} exists, and is seen to commute with all the elements of M; therefore, $\mathrm{Sdet}(M)$ is central in character. The comultiplication rule for the matrix representation of $GL_{p,q}(1|1)$, (4.3)–(4.5), preserves the bilinear product relations (4.5) and is parallel to the example of $GL_{p,q}(2)$ discussed earlier. The inverse of M is given by

$$M^{-1}=\begin{pmatrix} d^{-1} & -a^{-1}\beta a^{-1} \\ -d^{-1}\gamma d^{-1} & a^{-1}\end{pmatrix}\begin{pmatrix}\mathrm{Sdet}^{-1}(M) & 0 \\ 0 & \mathrm{Sdet}(M)\end{pmatrix}$$
$$=\begin{pmatrix}\mathrm{Sdet}^{-1}(M) & 0 \\ 0 & \mathrm{Sdet}(M)\end{pmatrix}\begin{pmatrix} d^{-1} & -d^{-1}\beta d^{-1} \\ -a^{-1}\gamma a^{-1} & a^{-1}\end{pmatrix} \tag{4.7}$$

where the inverse of the quantum superdeterminant is

$$\mathrm{Sdet}^{-1}(M)=da^{-1}+\beta a^{-1}\gamma a^{-1}. \tag{4.8}$$

From (4.6) it follows that $M^{-1}\in GL_{p^{-1},q^{-1}}(1|1)$. Computing M^n explicitly, for $n\in\mathbb{Z}$, it may be deduced that $M^n\in GL_{p^n,q^n}(1|1)$, as in the case of $GL_{p,q}(2)$.

The bilinear product relations (4.5) may be succinctly expressed in terms of a graded R-matrix condition for the quantum supergroup $GL_{p,q}(1|1)$:

$$R_{i_1i_2,k_2k_1}[(-1)^{(j_1+1)(k_2+j_2)}M_{k_1j_1}M_{k_2j_2}]=[(-1)^{(i_1+1)(i_2+k_2)}M_{i_2k_2}M_{i_1k_1}]R_{k_1k_2,j_2j_1} \tag{4.9}$$

adopting the $\mathbb{Z}_2$ grading factor 1 (-1) for an index i corresponding to the first (second) row or column of a 2×2 matrix space. The R-matrix is given by

$$R=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & p & 0 & 0\\ 0 & 1-pq & q & 0\\ 0 & 0 & 0 & pq\end{pmatrix} \tag{4.10}$$

where (i_1i_2) and (k_1k_2) label the rows and columns, respectively. In the limiting case $p=q$ the R-matrix in (4.10) reduces to the R-matrix corresponding to $GL_q(1|1)$ discussed by Schmidke *et al* (1990). The R-matrix in (4.10) satisfies the graded Yang–Baxter equation

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12} \tag{4.11}$$

where

$$\begin{aligned} R_{(12)i_1i_2i_3,j_3j_2j_1}&=R_{i_1i_2,j_2j_1}\delta_{i_3j_3}\\ R_{(13)i_1i_2i_3,j_3j_2j_1}&=(-1)^{(i_2+1)(i_3+j_3)}R_{i_1i_3,j_3j_1}\delta_{i_2j_2}\\ R_{(23)i_1i_2i_3,j_3j_2j_1}&=(-1)^{(i_1+1)(i_2+j_2+i_3+j_3)}R_{i_2i_3,j_3j_2}\delta_{i_1j_1}. \end{aligned} \tag{4.12}$$

In the Yang–Baxter equation we follow the matrix multiplication rule

$$(AB)_{i_1i_2i_3,j_3j_2j_1}=A_{i_1i_2i_3,k_3k_2k_1}B_{k_1k_2k_3,j_3j_2j_1}. \tag{4.13}$$

5. On the representations of the elements of $GL_{p,q}(1|1)$ matrices

The set of Heisenberg–Weyl variables for $GL_{p,q}(1|1)$ may be chosen as

$$\beta \qquad \gamma \qquad d \qquad \mathrm{Sdet}(M) \tag{5.1}$$

with the solution for the remaining element a of M given by

$$a = [\mathrm{Sdet}(M) + \beta d^{-1}\gamma d^{-1}]d. \tag{5.2}$$

As already noted d^{-1} must exist in order to define $\mathrm{Sdet}(M)$. To proceed further, we define the bosonic variables b and c such that

$$\beta = \theta b \qquad \gamma = \theta' c \tag{5.3}$$

where θ and θ' are constant Grassman numbers. Now we consider the variables (m_A):

$$m_1 = b \qquad m_2 = c \qquad m_3 = d \qquad m_4 = \mathrm{Sdet}(M). \tag{5.4}$$

Then, the representation procedure for $\mathrm{GL}_{p,q}(1|1)$ follows closely our analysis of $\mathrm{GL}_{p,q}(2)$ in section 3.

As before, taking, in general,

$$p = \exp(\mathrm{i}\chi) \qquad q = \exp(\mathrm{i}\lambda) \qquad 0 \leqslant \chi, \lambda < 2\pi \tag{5.5}$$

we have for (m_A) in (5.4)

$$m_A m_B = \exp(\mathrm{i}\phi_{AB}) m_B m_A \qquad \phi_{BA} = -\phi_{AB} \tag{5.6}$$

with

$$[\phi_{AB}] = \Phi = \begin{pmatrix} 0 & \lambda-\chi & \lambda & 0 \\ \chi-\lambda & 0 & \chi & 0 \\ -\lambda & -\chi & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.7}$$

For this Φ with rank 2

$$\varepsilon = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.8}$$

and a solution for U satisfying

$$\Phi = U\varepsilon U^{\mathrm{T}} \tag{5.9}$$

is given by

$$U = U^{(0)} = \begin{pmatrix} (\chi-\lambda)\sigma_1 & -\lambda & \sigma_1 & 0 \\ (\chi-\lambda)\sigma_2 & -\chi & \sigma_2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.10}$$

$$\chi\sigma_1 - \lambda\sigma_2 = 1.$$

Now, from the structure of ε in (5.8) it is clear that (m_A) can be realized in terms of a single pair $(\hat{Q}_1, \hat{Q}_2) = (\hat{P}, \hat{X})$. Then with the help of (5.10) we can write down the explicit representation of the elements of $\mathrm{GL}_{p,q}(1|1)$ as

$$\begin{aligned} &\beta \sim \theta \exp\{\mathrm{i}[(\chi-\lambda)\sigma_1 \hat{P} - \lambda \hat{X}]\} \\ &\gamma \sim \theta' \exp\{\mathrm{i}[(\chi-\lambda)\sigma_2 \hat{P} - \chi \hat{X}]\} \\ &d \sim \exp(\mathrm{i}\hat{P}) \\ &\mathrm{Sdet}(M) \sim 1 \end{aligned} \tag{5.11}$$

with a given by (5.2).

In the above realization (5.11) one can embed a group of continuous real parameters utilizing the invariance of the Heisenberg commutation relation $[\hat{X}, \hat{P}] = \mathrm{i}$ under the transformations $\hat{X} \to s_{11}\hat{X} + s_{12}\hat{P}$, $\hat{P} \to s_{21}\hat{X} + s_{22}\hat{P}$, $s_{11}s_{22} - s_{12}s_{21} = 1$ forming the group $\mathrm{Sp}(2, \mathbb{R}) \sim \mathrm{SL}(2, \mathbb{R})$. This would enable one to visualize a differential calculus on $\mathrm{GL}_{p,q}(1|1)$.

As in the case of $\mathrm{GL}_{p,q}(2)$ one can easily conclude that the above realization (5.11) is, in general, reducible by considering two cases: (i) both p and q are roots of unity, and (ii) p and q are not roots of unity but are commensurate. In these cases the representations of the elements of the $\mathrm{GL}_{p,q}(1|1)$ matrix can be written in terms of (μ_A) as follows:

$$\begin{aligned} &\beta \sim \theta \mu_1^{(\bar{l}-\bar{k})\bar{\sigma}_1/\bar{\varepsilon}} \mu_2^{\bar{l}/\bar{\varepsilon}} \mu_3^{\bar{\sigma}_1} \\ &\gamma \sim \theta' \mu_1^{(\bar{l}-\bar{k})\bar{\sigma}_2/\bar{\varepsilon}} \mu_2^{\bar{k}/\bar{\varepsilon}} \mu_3^{\bar{\sigma}_2} \\ &d \sim \mu_1 \qquad \mathrm{Sdet}(M) \sim \mu_4 \end{aligned} \tag{5.12}$$

where (μ_A) have the commutation relations of the form

$$\mu_1\mu_2 = \omega\mu_2\mu_1 \qquad \mu_A\mu_B = \mu_B\mu_A \text{ otherwise.} \tag{5.13}$$

When p and q are roots of unity as defined in (3.11), $\bar{k} = k$, $\bar{l} = l$, $\bar{\varepsilon} = \varepsilon$, $\omega = \exp(\mathrm{i}2\pi\varepsilon/N)$ with $\varepsilon = \gcd(k, l)$ and $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are integers $\in \mathbb{Z}_{N_1}$, $N_1 = N/(\gcd(N, \varepsilon))$, defined by $k\bar{\sigma}_2 - l\bar{\sigma}_1 = \varepsilon \pmod{N_1}$. When p and q are not roots of unity but are commensurate then with the same parametrization as in (3.26), $\bar{k} = k'$, $\bar{l} = l'$, $\bar{\varepsilon} = \varepsilon'$, $\omega = z'^{\varepsilon'}$, and $\varepsilon' = \gcd(k', l')$; now $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are any integers satisfying $k'\bar{\sigma}_2 - l'\bar{\sigma}_1 = \varepsilon'$. It is clear that regarding the matrix representations using (5.12) and (5.13) the same statements as for $\mathrm{GL}_{p,q}(2)$ hold; hence, the realization (5.11) is, in general, reducible. In the limiting case $p = q$, we have $\mathrm{GL}_q(1|1)$ and the representations in this case are obtained using the same formulae as above; for example, the $(\hat{X}, \hat{P})$ operator realization (5.11) now reduces to

$$\begin{aligned} &\beta \sim \theta \exp(-\mathrm{i}\lambda\hat{X}) \qquad \gamma \sim \theta' \exp(-\mathrm{i}\lambda\hat{X}) \\ &d \sim \exp(\mathrm{i}\hat{P}) \qquad \mathrm{Sdet}(M) \sim 1. \end{aligned} \tag{5.14}$$

6. On the representations of $A_q^{n|0}$ and $A_q^{*0|n}$

We shall now analyse the work of Floratos (1990) on the representation of $A_q^{n|0}$ in terms of quantum-mechanical phase space operators, using our approach. Let the commutation relations of the coordinates $(x_1, x_2, \ldots, x_n) \in A_q^{n|0}$ be

$$x_A x_B = q^{-1} x_B x_A \qquad A < B \qquad A, B = 1, 2, \ldots, n. \tag{6.1}$$

Following Floratos (1990) let us seek unitary realizations of (x_A) corresponding to unimodular value of q. Choosing $q=\exp(-i\lambda)$ $(0\leqslant\lambda<2\pi)$, (6.1) can be rewritten as

$$\begin{aligned} &x_Ax_B=\exp(i\phi_{AB})x_Bx_A \qquad \phi_{AB}=-\phi_{BA}=\lambda(1-\delta_{AB}) \\ &A<B \qquad A, B=1,2,\ldots,n. \end{aligned} \tag{6.2}$$

Now, the Φ-matrix, $[\phi_{AB}]$, associated with these commutation relations (6.2) may be written as

$$\Phi=U\varepsilon U^{\mathrm{T}} \tag{6.3}$$

where

$$\varepsilon_{AB}=\begin{cases} 1 & \text{for } A=2j-1, B=2j, j=1,2,\ldots,[n/2] \\ -1 & \text{for } A=2j, B=2j-1, j=1,2,\ldots,[n/2] \\ 0 & \text{otherwise} \end{cases}$$

$$u_{AB}=\begin{cases} \sqrt{\lambda} & \text{for } A=B=1,2,\ldots,n \\ 0 & \text{for } A<B \\ 0 & \text{for } A=2j, B=2j-1, j=1,2,\ldots,[n/2] \\ -\sqrt{\lambda} & \text{for } A=2j+k, k=1,2,\ldots,n-2j, B=2j-1, j=1,2,\ldots,[n/2] \\ \sqrt{\lambda} & \text{for } A=2j+k, k=1,2,\ldots,n-2j, B=2j, j=1,2,\ldots,[n/2]. \end{cases} \tag{6.4}$$

This leads to the realization

$$\begin{aligned} &x_1=\exp(i\sqrt{\lambda}\,\hat{P}_1) \qquad x_2=\exp(i\sqrt{\lambda}\,\hat{X}_1),\ldots \\ &x_{2j-1}=\exp\left(i\sqrt{\lambda}\sum_{k=1}^{j-1}(\hat{X}_k-\hat{P}_k)+\hat{P}_j\right) \\ &x_{2j}=\exp\left(i\sqrt{\lambda}\sum_{k=1}^{j-1}(\hat{X}_k-\hat{P}_k)+\hat{X}_j\right) \qquad j=1,2,\ldots,[n/2] \\ &x_n=\exp\left(i\sqrt{\lambda}\sum_{k=1}^{r}(\hat{X}_k-\hat{P}_k)\right) \qquad \text{if } n=2r+1 \end{aligned} \tag{6.5}$$

apart from unimodular multiplicative constants; here, $[n/2]$ stands for the integral part of $n/2$. The linear canonical transformations of the $2[n/2]$ operators $((\hat{X}_k,\hat{P}_k)|[\hat{X}_k,\hat{P}_l]=i\delta_{kl},\ [\hat{X}_k,\hat{X}_l]=[\hat{P}_k,\hat{P}_l]=0,\ k,l=1,2,\ldots,\ [n/2])$ forming the group $\mathrm{Sp}(2[n/2],\mathbb{R})$ provide a set of continuous real parameters for a concrete realization of a differential calculus on $A_q^{n|0}$, as is clear from the representation (6.5). This essentially reproduces the result of Floratos (1990) on the realization of the non-commutative coordinate space $A_q^{n|0}$ in terms of quantum-mechanical phase space operators giving an example of the way a differential calculus can be formulated in a non-commutative space by the introduction of continuous numerical parameters.

Exactly parallel to the situation obtained for $\mathrm{GL}_{p,q}(2)$ with p and q being roots of unity, the above $(\hat{X},\hat{P})$ operator realization of $A_q^{n|0}$ is not irreducible. With $(\mu_A|A=1,2,\ldots,n)$ defined by

$$\begin{aligned} &\mu_{2j-1}\mu_{2j}=\exp(i\lambda)\mu_{2j}\mu_{2j-1} \qquad j=1,2,\ldots,[n/2] \\ &\mu_A\mu_B=\mu_B\mu_A \text{ otherwise} \qquad A, B=1,2,\ldots,n \end{aligned} \tag{6.6}$$

one can write

$$
\begin{aligned}
&x_1 = \mu_1,\ x_2 = \mu_2, \ldots \\
&x_{2j-1} = \left(\prod_{k=1}^{j-1} \mu_{2k-1}^{\dagger}\mu_{2k}\right)\mu_{2j-1} \\
&x_{2j} = \left(\prod_{k=1}^{j-1} \mu_{2k-1}^{\dagger}\mu_{2k}\right)\mu_{2j} \qquad j = 1, 2, \ldots, [n/2] \\
&x_n = \prod_{k=1}^{r} \mu_{2k-1}^{\dagger}\mu_{2k} \qquad \text{if } n = 2r+1
\end{aligned}
\tag{6.7}
$$

so that irreducible representations of (x_A) are given in terms of the irreducible representation of (μ_A) obeying (6.6). To obtain (6.7) one has to just note that in this case the matrices $(P, \tilde{\varepsilon}, \tilde{U})$ are respectively given by $(\Phi/\lambda, \sim\lambda\varepsilon)$ and the same U as in (6.4). The irreducible representations of (μ_A) are of the form

$$
\begin{aligned}
&\mu_{2j-1} \sim 1\otimes 1\otimes \ldots \otimes 1 \otimes h \otimes 1 \otimes \ldots \otimes 1 \otimes 1 \\
&\mu_{2j} \sim 1\otimes 1\otimes \ldots \otimes 1 \otimes g \otimes 1 \otimes \ldots \otimes 1 \otimes 1 \\
&\text{for } j = 1, 2, \ldots, [n/2] \qquad \mu_n \sim 1 \text{ for } n = 2r+1 \qquad hg = \exp(\mathrm{i}\lambda)gh.
\end{aligned}
\tag{6.8}
$$

Thus, when $\lambda/2\pi$ is a rational number one has only finite-dimensional irreducible representations for (x_A) and the realization (6.5) is fully reducible uniquely as the sum of these irreducible representations. When $\lambda/2\pi$ is an irrational number then, as mentioned already, one would have an infinite number of irreducible representations for (x_A), all of infinite dimensions, and the representation (6.5) is reducible into these irreducible components but in an infinite number of different ways. If we want to restrict ourselves to only the irreducible representations provided by (6.7) and (6.8) then one can embed only a set of integer parameters in such a realization of (x_A); the set of integer parameters would correspond to an integer subgroup of $\mathrm{Sp}(2[n/2], \mathbb{R})$ depending on the particular value of q.

The representation of the dual space $A_q^{*0|n}$ with the coordinates $(\zeta_i | i = 1, 2, \ldots, n)$ satisfying the relations ($\zeta_i\zeta_j + q\zeta_j\zeta_i = 0$, $i < j$, $\zeta_i^2 = \zeta_j^2 = 0$, $i, j = 1, 2, \ldots, n$) follows closely the above analysis once the Grassman numbers are introduced as done above in the discussion of $\mathrm{GL}_{p,q}(1|1)$. To this end one can take $\zeta_j = \theta_j y_j$ with $\theta_j\theta_k + \theta_k\theta_j = 0$ so that (y_j) are bosonic coordinates obeying $y_j y_k - q y_k y_j = 0$, $j < k$, $j, k = 1, 2, \ldots, n$.

It is obvious that our approach to the representation of the Heisenberg–Weyl-type relations (1.1) can be used to obtain the realizations of more general non-commutative spaces with the coordinates (x_A) obeying the relations of the form $x_A x_B = q_{AB} x_B x_A$. Finally, it may be noted that the construction of (x_A) as above is a straightforward generalization of the process for $q = -1$ found in Clifford's fundamental paper (1878) on the application of Grassman's extensive algebra; there, Clifford shows that his $(2m+1)$-way geometric algebra, currently known as the Clifford algebra with $(2m+1)$ generators, is a compound of m quaternion algebras, the units of which are commutative with one another.

7. Conclusion

A quantum group may be defined as an endomorphism of an associated non-commutative coordinate space (Manin 1988, 1989). It turns out that one can have a formalism of differential calculus (Wess and Zumino 1990) on such a coordinate space which is fully covariant under the action of the corresponding quantum group. The application of non-commutative differential calculus on the defining matrix representation of a quantum group (Woronowicz 1987) leads to a deformation of the corresponding Lie algebra. Floratos (1990) studied the realization of the Manin quantum plane $A_q^{n|0}$, with $|q|=1$, in terms of the unitary phase space operators of quantum mechanics utilizing the Heisenberg–Weyl structure of the defining commutation relations of the non-commutative coordinates in $A_q^{n|0}$. For a generic unimodular q such a representation admits the embedding of a set of continuous real parameters corresponding to the symplectic group $\mathrm{Sp}(2[n/2], \mathbb{R})$ so that the scheme of a differential calculus on such a non-commutative space can be easily understood. It was, however, noted that if we restrict ourselves to irreducible finite-dimensional representations, e.g. when q is a primitive Nth root of unity, there is no such freedom to facilitate the formulation of a 'differential' calculus in $A_q^{n|0}$. We note that for a similar understanding of the differential calculus in the non-commutative space of the quantum matrices, the formulation of the representation of the elements of the quantum matrices in terms of the Heisenberg–Weyl type variables can be used; such a formalism exists already for $\mathrm{GL}_q(n)$ (Floratos 1989, Weyers 1990, Chakrabarti and Jagannathan 1991a). In this paper, we extend this formalism to the quantum group $\mathrm{GL}_{p,q}(2)$ and its superanalogue $\mathrm{GL}_{p,q}(1|1)$.

For $\mathrm{GL}_{p,q}(2)$, it is found that one can represent the elements of the quantum matrix in terms of exponentials of two mutually commuting pairs of canonically conjugate $(\hat{X}, \hat{P})$ operators. This allows the freedom of introducing in the realization a set of continuous real parameters labelled by the elements of the symplectic group $\mathrm{Sp}(4, \mathbb{R})$. For $\mathrm{GL}_{p,q}(1|1)$ a similar realization requires only a single $(\hat{X}, \hat{P})$ pair and, correspondingly, one has in this representation a set of continuous real parameters provided by the elements of $\mathrm{Sp}(2, \mathbb{R}) \sim \mathrm{SL}(2, \mathbb{R})$. These operator representations are not irreducible; when both p and q are roots of unity these representations are uniquely reducible into irreducible components of finite dimensions and when p and q are not roots of unity the irreducible representations are infinite-dimensional and the reduction of the operator realizations is not unique. But, if we restrict ourselves to the irreducible representations it is possible only to introduce a set of integer parameters provided by certain integer subgroups of the symplectic groups depending on p and q; with such irreducible representations one has only a 'lattice differential' calculus on the given algebraic structure.

It is noted that, unlike the elements of the Heisenberg–Weyl group or the generators of generalized Clifford algebras, all the Heisenberg–Weyl-type variables associated with the elements of a quantum matrix are not necessarily invertible. This leads to the possibility of several inequivalent finite-dimensional representations of the elements of the quantum matrices of $\mathrm{GL}_{p,q}(2)$ and $\mathrm{GL}_{p,q}(1|1)$. One can also have operator realizations of this type with the facility to introduce a set of continuous parameters; but, here again, the finite-dimensional representations admit only integer parameters. Finally, we have analysed the work of Floratos (1990) on the representations of the Manin hyperplane, using the Heisenberg–Weyl relations, within the framework of our method of realization of such relations. Essentially, following Floratos (1990) we have provided examples of realizations for $\mathrm{GL}_{p,q}(2)$ and $\mathrm{GL}_{p,q}(1|1)$ which, albeit reducible, contain sets of continuous real parameters enabling one to visualize a formalism of differential calculus on the corresponding non-commutative spaces; for $\mathrm{GL}_q(n)$ such realizations are obtainable from earlier work (Floratos 1989, Weyers 1990, Chakrabarti and Jagannathan 1991a).

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